Averaging Over the Unitarian Group and the Monotonicity Conjecture of Merris and Watkins by Avital Frumkin

abstruct

We show that the monotonicity conjecture of Merriss and Watkins is true in average when taking the set of matrices of given non negative spectra as probability space with respect to the Haar measure of the unitarian group .

1.   Introduction

Let $\mathbb{C}S_n$ be the group algebra of the symmetric group S_n over the complex numbers. ,$\mathbb{C}S_n$ includes all the functions from S_n to $\mathbb{C}$ as vector space of dimension N! . The multiplication is defined as convolution of group's functions , i.e.

$$f \cdot g(\sigma) = \sum_{\gamma \in S_n} f(\gamma) \ g(\gamma^{-1}\sigma) \tag{1}$$

Given $n \times n$ complex matrix A, we define a function from $\mathbb{C}S_n$ to $\mathbb{C}$ by the formula

$$f \in \mathbb{C}S_n \qquad f \to d_f(A) = \sum_{\sigma \in S_n} f(\sigma) \prod_i A_{i \ \sigma(i)} \tag{2}$$

(d is for "determinant")

In these notations, the determinant of a matrix A is given by the formula

$$det A = d_f(A), f : S_n \to \mathbb{C}, f(\sigma) = sig(\sigma). \tag{3}$$

The permanent of A is

$$d_f(A), f : S_n \to \mathbb{C}, f(\sigma) = 1. for any \sigma \in S_n. \tag{4}$$

For given irreducible characterχ of S_n,after identifying it as a function from $\mathbb{C}S_N$to the complexes $d_\chi(A)$ is called the immanent of A corresponding with χ.

a partition of N is a vector of integers $\eta = \eta_1, \eta_2 \cdots$ so that $\eta_1 \geq \eta_2 \cdots \geq 0$ and $\sum \eta_1 = N$. We denote η A partition of N by $\eta \vdash N$.

The irreducible characters of S_N correspond to partitions of N, hence each irreducible character is attached to a partition, . So we denote $d_\eta(A)$ instead of $d_{\chi_\eta}(A)$.

The first theorem in the area of immanents is that of Schur in 1918 . One corollary of it is that

$$det(A) \leq \frac{d_\eta(A)}{\chi(e)} \tag{5}$$

to any $n \times n$ Hermitian positive semi-definite matrix A and $\eta \vdash n$ once e is the trivial element of S_n.

In my view,even the fact that $d_\eta(A)$ is non-negative to any non-negative definite matrix A and $\eta \vdash n$ is amazing.

The word immanent, apparently invented by Littlewood,to these complex functions involving S_n group algebra and $n \times n$ matrices over $\mathbb{C}$. See Chapter (6) in his book The Theory of Group Character [Li]. Littlewood includes in his definition of immanent almost any monomials in the $n \times n$ matrices elements in his celebrated postulate about the relationships between immanents and Schur functions. Here is the postulate ".... corresponding to any relation between Schur functions of total weight N, we may replace the Schur function by the corresponding immanents of complementary coaxial minor of $[a_{st}]$ provided that every product is summed for all sets of complementary coaxial minors."

This postulate, together with Lieb's inequality [Li], is the corner stone of the paper of Merris and Watkins [M W]. In their paper, they

used it in innovative way accomplishing inequalities and identities of immanents and matrix functions

Among other things, they raised a conjecture which was called afterwards by Pate [P3] the Merris and Watkins (monotonicity) conjecture . For treating it, we need some definitions.

For a given partition of $N, \eta = \eta_1 \geq \eta_2 \geq \eta_n \geq 0$. The $\eta's$ weight space of $\otimes^N \mathbb{C}^n$ is spent by all the tensors $v_{i_1} \otimes \cdots \otimes v_{i_N}$ for which the number of occurrences of v_j, i.e. the j basis vector of $\mathbb{C}^n$, is η_j. The weight spaces in $\otimes^N \mathbb{C}^n$ are S_N module with respect to the Schur action of S_N on $\otimes^N \mathbb{C}^n$: for $\sigma \in S_n$ its Schur action is defined by the formula

$$\sigma(v_{i_1} \otimes \cdots \otimes v_{i_N}) = v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(N)}} \tag{6}$$

. The S_N's character corresponding with this η weight space is denoted $[\eta]$ in [M W]

In the language of character theory, $[\eta]$ is the induce character from the trivial character of the Young sub group corresponding with η see [J K]to S_N

The dominant order on the set of partitions of $N \succ$,is defined by the formula

$$\eta \succ \eta' \Leftrightarrow \sum_{i=1}^{j} \eta_i \geq \sum_{i=1}^{j} \eta_i' \quad \text{to any } j \tag{7}$$

This order between partitions of N has an intimate connection to the action of the Lie algebra $g\ell_n(\mathbb{C})$ on $\otimes^N \mathbb{C}^n$ and through it with the action of $GL_n(\mathbb{C})$ on $\otimes^N \mathbb{C}^n$.

By using a clever result in representation theory of symmetric groups (theorem 22 of [JK])(among other results), Merris and Watkins proved that for any non-negative definite $n \times n$ matrix A

$$\eta \succ \eta' \Rightarrow d_{[\eta]} A \leq d_{[\eta']} A \tag{8}$$

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On the other hand , they conjectured (God know how)that

$$\binom{n}{\eta}^{-1} d_{[\eta]} A \geq \binom{n}{\eta'}^{-1} d_{[\eta']} A \ . \quad \binom{n}{\alpha} \quad \text{is the multinomial coef-}$$

ficient attached to the partition α.

Clearly $\binom{n}{\alpha}$ is the dimension of the $\alpha's$ weight space in $\bigotimes^n \mathbb{C}$.

We shall prove (theorem 7)that when $\eta \succeq \eta'$

$$\binom{n}{\eta}^{-1} \int d_{[\eta']} A^u du \geq \binom{n}{\eta'}^{-1} \int d_{[\eta']} A^u du. \quad \text{when } A^u = uAu^{-1} \text{ and}$$

the integration is done over the Unitarian group U_n with respect to the Haar measure du of U_n .

When n tends to infinity, and η is a partition of n of a bounded number of parts, the character$[\eta]$

tends to be closer to χ_η, the irreducible character corresponding with η, in a reasonable matric.More precisely when $[\eta]$ is written as sum of ireduceabl characters the contribution of those characters far from χ_η to the dimension of $[\eta]$ vanishes in comparison to the full dimension. See [F.G] or [M.H.Ch](under the tittle Kyel Werner theorem)

Because of that and the fact that the induce characters are easier than the irreducible one to compute, the asymptotic behavior of them is worth attention when looking for counter examples.Indeed this is how I got to consider them . Hence before turning to the Merris and Wotkins monotonicity conjecture in average , we survey the monotonicity of immanents' status as I know it

I believe that the next theorem of Pate cover the scope till now
Given a partition $\eta = \eta_1 \geq \eta_2 \geq \cdots \eta_i > \eta_{i+1} \cdots$ let $\eta' = \eta_1 \geq \eta_2 \geq \cdots \eta_i - 1 \geq \eta_{i+1} \cdots \geq 1 \Rightarrow \frac{d_{\eta'}(A)}{\chi_{\eta'}(e)} \leq \frac{d_\eta(A)}{\chi_\eta(e)}$.
That is to say that in removing a corner in the Young diagram of shape

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η to the end decreases the normalized immanent corresponding with the new partition.See [P2] , The first large step in this direction was done in [P 1]

Our approach to computations of the S_n matrix function begins with an observation that appeared in [Kos] where Kostant reproved that

$d_\eta(A) > d_\eta(I)$ for non-negative definite, or totally positive matrices,A of determinant 1 .

His observation is that $d_\eta(A)$ is the "trace" of $A^{\otimes n}$ when it acts on $M_\eta(0)$. The zero weight subspace of M_η where M_η is the irreducible $GL_n(\mathbb{C})$ module corresponding with η. The zero weight space is the subspace of all the Tori invariant vectors. In other words the subspace of the equipartition weight . The parenthesis over the trace was needed since $M_\eta(0)$ is not respected by $A^{\otimes n}$ for a general matrix A and some projection is needed. But by averaging the immanent of uAu^* over the Unitarian group with respect to its Haar measure , the projection, can be ignored

Given $f \in \mathbb{C}S_N$ we denote $\hat{d}_f(A) = \int d_f(A^u)du$ integrated over the unitarian group with respect to the Haar measure

Let us denote $\hat{d}_\eta(A)$ for $\hat{d}_{\chi_\eta}(A)$. We shall prove (theorem 5)

$$\hat{d}_\eta(A) = trace \int A^{u \otimes n} du \Big|_{M_{\eta(0)}} = \frac{s_\eta(A)}{s_\eta(I)} dim(\ M_\eta(0)) \qquad (9)$$

By $s_\eta(\mathcal{A})$ we mean the value of the Schur function corresponding with η under substitution of the spectra of $\mathcal{A}$. An important fact is that $dim\ M_\eta(0) = \chi_\eta(e)$, i.e., the dimension of the S_n irreducible module corresponding with η. See [Kos]or lemma 5.

It is worth attention that formula 9 for $\hat{d}_\eta(A)$ is the expected con-

tribution of a random subspace $M_\eta(0)$

to the trace of a random operator when acts on a whole space M_η, when using an orthogonal form to compute the trace.i e this is the most likely number one wold evaluate $d_\eta(A)$to A of given spectra Hence through the monotonicity theorem we have proof the Merris Watkin monotonicity conjecture is definitely supported since the averaging values we compute are seemed to be generic .

one can naturally ask about averaging over the Orthogonal group in place of the Unitarian group. The answer is quite complicated since $A^{\otimes n}$ usually even doesn't respect the irreducible modules of the Orthogonal group So first of all the problem has to be defined more delicately . Next one has to challenge the difficulties which the Brower algebra produces . Hopefully it will be treated elsewhere

Now the identity 9 enables us to prove(lemma 7) that for any S_n's submodule $V \subseteq \otimes^n \mathbb{C}^n, \hat{d}_{\chi_V}(A) = trace(\int A^{u \otimes n}) |_V$

where χ_V is the character of S_n action on V. The trace is defined since such integrals respect S_n's submodule.After this is done , to prove the averaging version of Merris and Watkins conjecture is a matter of some explicit traces computations.(theorem 7)

Sections 2,3,4 can be thought of as preliminaries in representation theory relevant to our treatment afterwards . Sections 5,6 deal with special central element operators on $\otimes^n \mathbb{C}^n$ given by integrations over the Unitarian group. In Section 7 we prove the monotonicity result.

Our treatment doesn't use any explicit integration over the unitarian group .On the contrary , in section 8 we bring a formula for the average of multiplicity-free products of matrix-elements, in terms of Schur-functions and characters of the symmetric group. The average is being taken over the unitarian group with respect to its action by conjugation on the matrix (theorem 9).

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2. The tensor space $\otimes^N \mathbb{C}^n$ as $S_N(U_n)$ module

Let $v_1, v_2 \cdots v_m$ be an orthonormal basis. Denote the form by $\langle \, \rangle$.

Given N we extend the form to $\otimes^N \mathbb{C}^n$ by the formula

$$\langle v_{i_1} \otimes \cdots \otimes v_{i_N} \cdot v_{j_1} \otimes \cdots \otimes v_{j_N} \rangle = \prod_k \langle v_{i_k} v_{j_k} \rangle \qquad (10)$$

Define the Schur action of the symmetric group S_N on $\otimes^N \mathbb{C}^n$ by the formula

$$\sigma(v_{i_1} \otimes \cdots \otimes v_{i_N}) = v_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes v_{i_{\sigma^{-1}(N)}} \qquad \sigma \in S_N \qquad (11)$$

For a given $n \times n$ matrix A, define the diagonal action of A on $\otimes^N \mathbb{C}^n$ by the formula

$$A(v_{i_1} \otimes \cdots \otimes v_{i_N}) = A v_{i_1} \otimes \cdots \otimes A v_{i_N} \qquad (12)$$

Some times we call the operator coresponding with A" the N Kronecker power of A" and denote it by $A^{\otimes N}$

A monotonic vector of integers $\bar{\eta} = \eta_1 \geq \eta_2 \geq \cdots \eta_N$ is called a partition of N ; $\eta \vdash N$ if $\Sigma \eta_i = N$.

For $\eta \vdash N$ define the η weight space in $\otimes^N \mathbb{C}^n$; $\otimes^N \mathbb{C}^n(\eta)$ by the formula

$$\otimes^N \mathbb{C}^n(\eta) = span[v_{i_1} \otimes \cdots \otimes v_{i_N}; \#[e : i_e = j] = \eta_j] \qquad (13)$$

For a subspace M of $\otimes^N \mathbb{C}^n$ we define

$$M(\eta) = M \cap \otimes^N \mathbb{C}^n(\eta) . \qquad (14)$$

The weight space of $\bar{\eta} = \eta_1 = \eta_2 = \eta_3 \cdots$ is called the zero weight space ; $M(0)$. In this note usually N=n so $\eta_i = 1$ in the last definition to zero weight spaces .

Now as it can easily be seen, the action of the symmetric group (11) and that of the Unitarian group(12) on the tensor product spaces commute with each other so they respect the isotypical (see section 3) component of one another . In fact they have an isotypical component in common in their action on $\otimes^N \mathbb{C}^n$ .So it is enough to treat just the isotypical component of the symmetric group.

because of the irreducible representations of S_N indexed by partitions of N it is reasonable to denote the isotypical components of $S_N(U_n)$ in$\otimes^N \mathbb{C}^n$ with partitions of N and identify such a component with $V_\eta \otimes M_\eta$ where V_η is S_N irreducible and $M_\eta$is U_n irreducible.

The Schur Wyel duality theorem shows this relationships between S_n and U_n representations on the tensor product spaces.

Theorem 1. *As $S_N(U_n) module \otimes^N \mathbb{C}^n$ isomorphic to $\bigoplus V_\eta \otimes M_\eta$ The sum goes over $\eta \vdash N$ of no more than n parts*

Proof can be found in [G W]

In the following sections we treat the action of the symmetric group on$\bigotimes^N \mathbb{C}^n$ to get more explicit expression of the isotypical component in the light of theorem 1 . We shall first give some basic information on representation theory .

3.   representation theory

Let G be a finite group and χ an irreducible character of it. Let $\mathbb{C}G$ be the group-algebra of G over $\mathbb{C}$. Define a central element in $\mathbb{C}G$ corresponding with χ by the formula

$$C_\chi = \frac{\chi(e)}{|G|} \sum_{g \epsilon G} \chi(g) \; g \tag{15}$$

$C\chi$ is central in $\mathbb{C}G$ because the characters of G are conjugacy invariant .More than this.

Lemma 1. *Given irreducible character χ, C_χ is idempotent and $\mathbb{C}G \cdot C_\chi$ is the isotypical component of $\otimes^N \mathbb{C}^n$ corresponding with χ.*

To prove this one use the orthogonality relations of the irreducible characters of a finite group. See [J.K].

For our treatment we formulate it more generally in the next lemma

Lemma 2. *Let M be a $G's$ module over $\mathbb{C}$ and χ be an irreducible character of G. Then MC_χ is the isotypical component of M corresponding with χ.*

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Along this note we exchange freely $GL_n(\mathbb{C})$ with the unitarian group U_n thanks to the next statement.

Lemma 3. *Each irreducible $GL_n(\mathbb{C})$ module remains irreducible under reduction to the unitarian group U_n and vice versa, i.e. each U_n's irreducible module occurs as a reduction from $GL_n(\mathbb{C})$ irreducible module See [GW] page 94.*

By " isotypical component" (of a module) we mean a maximal submodule with no non isomorphic submodules in it. The importance of the isotypical component a module is that intertwining operators i e operators which commute with the action of the group on the module respect it
"complete reducibility" is cleared by the next theorem

Theorem 2. *Let G be a compact group (may be finite) and let V be a G submodule over $\mathbb{C}$ than if U is V submodule there exists U' a V submodule so that $V = U \bigoplus U'$*

For proof see [G W]

4. Partitions, Young diagrams and Young tableaux and the construction of the $U_n \times S_N$ isotypical modules

For each partition of η of N, $\eta \vdash N$ one corresponds irreducible character of S_N χ_η. The explicit construction of χ_η is not needed for our treatment . A good reference for characters χ_η and their irreducible module V_η, is [J.K] or [G W].

Let $\eta \vdash N, \eta = \eta_1 \geq \eta_n \cdots \geq \eta_n$. For our discussion any $\eta \vdash N$ is no more than n part. See Schur Weyl duality(theorem 1)

Now the Young diagram of shape $\eta \vdash N$ is an array of rows of cells, one under the other, η_i cells are in the i row.

The rows begin together from the very left to the right, so one gets an array of columns from left to right as well . See the figure below.

Given N, a Young's tableaux of shape $\eta \vdash N$ is a filling of the cells of the Young's diagram by the numbers $1, 2 \cdots n$ so they increase down the column and non-decrease in the rows to the right.

Next corresponding to partition $\eta \vdash N$ of no more than n parts we construct a basis for an irreducible U_n module in $\otimes^N \mathbb{C}^n$

Given Young tableaux of shape η, we construct a vector in $\otimes^N \mathbb{C}^n$ by the next process. First, we fix an order on the cells in the Young diagram of shape η. Next, we attach the digits in the Young tableaux to the basis vectors of $\mathbb{C}^n$ one after another, with respect to the order we had fixed , along a tensor of length N

Let us take an example .

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 \\ \hline \end{array} \longrightarrow v_1 \otimes v_1 \otimes v_2 \otimes v_2 \otimes v_3 \ .$$

The order we fixed on the cells of the Young diagram of shape $\eta = 2 \geq$

$2 \geq 1$ is that we begin from the upper left down along the rows one after another ending at the lower right.

Now we act on the tensor we obtained with the central idempotent $C\chi_\eta$

Recall M_η denotes the U_n's irreducible module corresponding with $\eta \vdash N$.

Theorem 3. *(i) Fix an order on the cells of the Young diagram of shape η.*

A basis for M_η's copy, in $\otimes^N \mathbb{C}^n$ is accepted when using the process above over all the Young tableaux of shape η.

(ii) By continuing the process above in all the orders on the Young diagram of shape η cells, and over all the Young tableau of this shape one gets a generating set for the isotypical U_n component(and S_N as well)of type η.

See [K J] or [G W] for proof

Corollary

$dim(M_\eta)$ is the number of Young tableaux of shape η.

Let us define for $\eta \vdash N$ a standard Young tableau as a filling of the Young diagram of shape η with the letters $1\ 2\cdots N$ so that they increase down the column and in the rows to the right.

Theorem 4. $dim(V_\eta)(\chi_\eta(e))$, *is the number of standard Young tableaux of shape η.*

Proof. See [J.K] [G W]

5.   Central $S_N \times U_n$ operators

Let A be a $n \times n$ matrix. Define an operator E_A^N on $\otimes^N \mathbb{C}^n$ by the formula

$$E_A^N = \int (A^u)^{\otimes N} du \tag{16}$$

integrated on the unitarian group with respect to the Haar measure
For $u \in U_n$ $A^u = uAu^*$.

Lemma 4. *The operator E_A^N is $S_N(U_n)$ equivariant.*

Proof. It is S_N equivariant since for each u $A^u \otimes^N$ is S_N equivariant. It is U_n equivariant since the Haar measure is U_n invariant.

   Now because E_A^N is $S_N(U_n)$ commute, it acts scalarly on the isotypical $S_N \times U_n$ components in $\otimes^N \mathbb{C}^n$.

   Let $s_\eta(A)$ denote the value of the Schur function $s_\eta(\bar{x})$ when the vector of the eigenvalues of A is substituted.

Lemma 5. $E_A^N \big|_{V_\eta \times M_\eta} = \frac{S_\eta(A)}{S_\eta(I)}$ *times the identity. I is the $n \times n$ unit matrix.*

Proof.   Because E_A^N is $S_N \times U_n$ equivariant it acts scalarly on the isotypical component, i.e. $V_\eta \times M_\eta$. By definition $s_\eta(A) = trace(A) \big|_{M_\eta}$; $s_\eta(I) = dim(M_\eta)$. Hence the scalar is $\frac{s_\eta(A)}{s_\eta(I)}$.

Remark E_A^N depends only on the spectrum of A, since the traces of A on U_n's modules depends only on the spectra of A.

   It is because the trace of A on each $GL_n(\mathbb{C})$ module depends just on $A's$ spectra, see [FG].

6.   The traces of E_A^n on some subspaces of $\otimes^n \mathbb{C}^n$

Recall that $M_\eta(0)$ is the intersection of M_η with the zero weight space of $\otimes^n \mathbb{C}^n$.

   By the last lemma $trace E_A^n \big|_{M_\eta(0)} = dim(M_\eta(0)) \frac{s_\eta(A)}{s_\eta(I)}$.

Lemma 6. $dim M_\eta(0) = dim(V_\eta)$

Proof. Recall how a basis to M_η was constructed through Young tableaux.(theorem 3) Now the Young tableaux corresponding with zero weight tensors are those on which each digit from $12\cdots n$ appears once. Such a Young tableaux is standard. Hence by Theorem 4 the lemma is proved.

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Corollary $trace\, E_A^N\big|_{M_\eta(0)} = \frac{s_\eta(A)}{s_\eta(I)}\chi_n(e)$.

RemarkIf one takes $m \leqslant n$ and $M_\eta(\gamma)$ for $\gamma \vdash m$ which is multiplicity free weight, the last lemma remain true (we shall use it in section 8)

Now we are going to compute $trace\, E_A^N\big|_{M_\eta(0)}$ explicitly by using the standard basis of $\otimes^n\mathbb{C}^n(0)$ and the central idempotent $C_\eta = \frac{\chi_n(e)}{n!}\sum_{\sigma\in S_n}\chi_\eta(\sigma)\sigma$.(Formula 15) This will be done along the proof of the next theorem.

We define $\hat{d}_\eta(A) = \int d_\eta A^u du$ integrated over the unitarian group with respect to the Haar measure on it.

Theorem 5. $trace E_A^n\big|_{M_\eta(0)} = \hat{d}_\eta(A)$

Proof $V_\eta \otimes M_\eta = \otimes^N\mathbb{C}^n \cdot C_\eta$. Hence $V_\eta \otimes M_\eta(0) = \otimes^N\mathbb{C}^n(0) \cdot C_\eta$.

Hence

$$trace E_A^n\big|_{M_\eta(0)} = \int du \sum_{\sigma\in S_n} \langle A^u v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \cdot C_\eta v_{\sigma(1)} \otimes \cdots v_{\sigma(n)}\rangle$$

$$= \int du\, n!\langle A^u v_1 \otimes \cdots \otimes v_n \cdot C_\eta v_1 \otimes \cdots \otimes v_n\rangle \qquad (17)$$

Here we use the fact that $\sigma C_\eta \sigma^{-1} = C_\eta$.

Now by injection of the explicit expression of C_η we come to the

last expression

$$\int du\ n!\frac{\chi(e)}{n!}\sum_{\sigma\in S_n}\chi_\eta(\sigma)\langle A^u v_1\otimes\cdots\otimes v_n\cdot v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}\rangle$$

$$=\chi_\eta(e)\int\sum_{\sigma\in S_n}\chi_\eta(\sigma)\prod_i A^u_{i\sigma(i)}du=\chi_\eta(e)\hat{d}_\eta(A)\qquad(18)$$

At the last step we used the multiplication formula to the scalar form (Formula 10). The proof ends since $\chi_\eta(e)$ is the multiplicity of $M_\eta(0)$ in $\otimes^N\mathbb{C}^n$.

To illustrate the last theorem we prove briefly the next theorem of Merris and Watkins ([M W]theorem 8).

Theorem 6. *Let A be a $n\times n$ matrix of rank k, then $d_\eta(A)=0$ to any $\eta\vdash n$ but if η has no more than k parts.*

Proof.

Recall s_η is the Schur function corresponding with the partition η Now $s_\eta(A)=0$ but if η has no more than k parts.

Indeed without loss of generality one can assume that A is a diagonal matrix (see the remark at the end of section 5).Now consider the action of $A^{\otimes n}$ on each vector basis corresponding with Young tableaux as in Theorem 3.If η has more parts than the rank of A this action vanishes identically

Now

$$s_\eta(A)=0\Rightarrow\hat{d}_\eta(A)=tracE^n_A\big|_{V_\eta\otimes M_\eta}=0\qquad(19)$$

Since for non-negative definite A $d_\eta(A^u)\geq 0$ to any $u\in U_n$ it is to say that $d_\eta(A)=0$. To prove the theorem, one can check that the non-negative matrices of rank k are Zarisky dense in the set of matrices of rank k.Thanks to A Goldberger for the remark

The next lemma is a clear corollary of theorem(5)and is pivotal to the proof of the monotonicity theorem

Lemma 7. *Let $V \subseteq \otimes^n \mathbb{C}^n$ be S_n's modules. Let χ_V be its character, i.e. $\chi_V(\sigma) = trace\ \sigma|_V$. Then*

$$trace E_A^N|_V = \sum_\sigma \chi_V(\sigma) \int \prod A_{i\sigma i}^u \qquad (20)$$

Proof. Because of the complete reducibility of any S_n's module over $\mathbb{C}$ and the linearity of the traces and the matrix functions d_f one can deal merely with irreducible modules.

Assume V is irreducible, let us say of type V_η. So $V \subseteq V_\eta \otimes M_\eta$ on which E_A^n acts scalarly, hence the trace of it depends just on its dimension Hence the lemma is proved using theorem (5)and lemma (6)

7.   The Merris Watkins monotonicity conjecture

As S_n module $\otimes^N \mathbb{C}^n(\eta)$, the submodule of tensors of weight η , isomorphic to the S_N module induced from the trivial module of the Young subgroup corresponding with η ,$S_{\eta_1} \times S_{\eta_2} \cdots$,. Let say $Young(\eta)$
i.e $1_{\eta_1} \otimes 1_{\eta_2} \cdots 1_{\eta_n} \bigotimes_{\mathbb{C}(Young(\eta))} \mathbb{C}S_n$.
Merris and Watkins denote the character of this S_n module by $[\eta]$.

Denote by $\succ$ the next relation on the partitions of n.

We say that $\eta \succ n'$ if $\sum_{i=1}^{j} \eta_i \geq \sum_{i=1}^{j} \eta'_i$ to any j. Merris and Watkins conjecture is that $\eta \succ \eta' \Rightarrow \frac{d_{[\eta]}A}{d_{[\eta]}I} \geq \frac{d_{[\eta']}A}{d_{[\eta']}I}$ for any non-negative definite matrix A.one can check that $d_{[\eta]}I = \begin{pmatrix} n \\ \eta \end{pmatrix} = \frac{n!}{\prod_i \eta_i!}$.

We prove that their conjecture is true when averaging with respect to conjugacy relation over U_n, the unitarian group.

Theorem 7. *Under the assumptions above*
$\eta \succ \eta' \Rightarrow \begin{pmatrix} n \\ \eta \end{pmatrix}^{-1} \hat{d}_{[\eta]}(A) \geq \begin{pmatrix} n \\ \eta' \end{pmatrix}^{-1} \hat{d}_{[\eta']}(A)$ *to any non-negative definite matrix A.*

The proof includes a series of reductions.

First by lemma (7) for $\gamma \vdash n$, of n

$$\hat{d}_{\lceil}\gamma]A^u = tracE_A^n \mid_{\otimes^N \mathbb{C}^n(\gamma)}$$

$$= \sum \int \langle v_{i_1} \otimes \cdots v_{i_n} A^{u^{\otimes n}} \cdot v_{i_1} \otimes \cdots \otimes v_{i_n} \rangle \, du \qquad (21)$$

summed over the tensors of weightγ

Now by using of the product form for the scaler product formula over the tensor space (10) we get for (21) the next elagant formula

$$\binom{n}{\gamma} \int \prod_i (A^u)_{ii}^{\gamma_i} \, du \qquad (22)$$

using the invariancy of the Haar measure to permutations conjugation we get the next formula

$$\binom{n}{\gamma} \int \sum_{\sigma \in S_n} \frac{1}{n!} \prod_i \left(A^u_{\sigma(i)\sigma(i)} \right)^{\gamma_i} \qquad (23)$$

Since A is non negative definite the diagonal elements of A^u are non negative hence the first reduction is to observe that it is enough to prove that for non-negative $b_1 b_2 \cdots b_n$

$$\sum_{\sigma \in S_n} \prod_i b_{\sigma(i)}^{\eta_i} \geq \sum_{\sigma \in S_n} \prod_i b_{\sigma(i)}^{\eta_i'} \qquad (24)$$

Next we observe that

$$\sum_{\sigma \in S_n} \prod_i b_{\sigma(i)}^{\gamma_i} = Perm\left(b_j^{\gamma_i} \right) \qquad \text{(Perm for permanent)} \qquad (25)$$

and hence then we will prove

$$Perm\left(b_j^{\eta_i} \right) \geq Perm\left(b_j^{\eta_i'} \right) \qquad (26)$$

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The next reduction is to assume that $\eta_i = \eta_i'$ to $i \geq 2$. We use the invariancy of the permanent we exchange rows as well as the fact that if $\eta \succ \eta'$ there exists a path of partitions between η to η' monotonic with respect to $\succ$ so that each consecutive partition along the path differ just on two parts.

Next we develop the last permanent in respect to their first two rows. We show that in each summand in the computation we get the desired inequality so the last reduction is to prove the next lemma.

Lemma 8. *Let A, B be positive and for a given n denote*
$$\varphi_{A,B}(x) = per \begin{pmatrix} A^x & B^x \\ A^{n-x} & B^{n-x} \end{pmatrix} \text{ then for } x \geq \tfrac{n}{2} \ \varphi_{AB}(x) \text{ increases with}$$
x

Proof.

We consider $\frac{d}{dx}\varphi_{AB}(x)$

$$\frac{d}{dx}\varphi_{AB}(x) = \frac{d}{dx}(A^x B^{n-x} + A^{n-x} B^x) = A^x B^{n-x} \log \frac{A}{B} + A^{n-x} B^x \log \frac{B}{A}$$

$$= [A^x B^{n-x} - B^x A^{n-x}] \log \frac{A}{B} = A^x B^{n-x} \left(1 - \left(\frac{B}{A}\right)^x \left(\frac{A}{B}\right)^{n-x}\right) \log \frac{A}{B}$$

now if $A \geq B$ then $\log\left(\frac{A}{B}\right) > 0$ and $\left(\frac{B}{A}\right)^x \left(\frac{A}{B}\right)^{n-x} \leq 1$ since $x \geq \tfrac{n}{2}$. Hence the derivation is positive . The other case is treated the same way.

This ends the proof of theorem (7).

As an example of using the monotonicity theorem, we give a proof to the theorem of James and Lieback [JL] on the dominancy of the permanents among the immanents of no more than two parts, but in average.

Theorem 8. *Let $\eta_1 \geq \eta_2$ be a partition of two parts then $\frac{\hat{d}_\eta(A)}{d_\eta(I)} \leq per(A)$ for any non-negative definite $n \times n$ matrix.*

Proof. As it is well known [JL], [JK], $\chi_\eta = [\eta] - [\eta']$ when $\eta' = \eta'_1 \geq \eta'_2$ so that $\eta'_1 = \eta_1 + 1$ so

$$\frac{\hat{d}_{[\eta]}(A)}{d_{[\eta]}(I)} = \frac{\hat{d}_{[\eta']}(A) + \hat{d}_\eta(A)}{\hat{d}_{[\eta']}(I) + d_\eta(I)} = \frac{\frac{\hat{d}_{[\eta']}(A)}{d_{[\eta']}(I)}d_{[\eta']}(I) + \frac{\hat{d}_\eta(A)}{d_\eta(I)}d_\eta(I)}{d_{[\hat{\eta}']}(I) + d_\eta(I)} \quad (27)$$

The last expression is a convex sum of $\frac{\hat{d}_{[\eta']}(A)}{d_{[\eta']}(I)}$ and $\frac{\hat{d}_\eta(A)}{d_\eta(I)}$. Hence it is greater than the minimum. Assume $\frac{\hat{d}_\eta(A)}{d_\eta(I)} > per(A) \geq \frac{\hat{d}_{[\eta']}A}{d_{[\eta']}(I)}$ one gets $\frac{\hat{d}_{[\eta]}(A)}{d_{[\eta]}(I)} > \frac{\hat{d}_{[\eta']}(A)}{d_{[\eta']}(I)}$ and this contradicts the monotonicity we have just proved.

Remark on generalizations

One can check that under the next definition of

$\hat{d}_\eta(A)$ for $\eta \vdash$ m $\leqslant n$ theorem 7 remain true

The definition is given in the next formula

For $m \leqq n$ and $\eta \vdash m$

$$\hat{d}_\eta(A) = \int \sum_{\sigma \in S_m} \chi_\eta(\sigma) \prod_{i \leq m} A^u_{i\sigma(i)} du \quad (28)$$

integrated over the unitarian group U_n

Especialy lemma 6 remain true for $m \leqq n$.

With this last remark we are coming to the last section dealing with some explicit expressions for the integration of matrix monomials which have occurred over the note

8. U_n Invariant matrix's elements' products

Let A be $n \times n$ complex matrix and $\lambda_1, \lambda_2...\lambda_n$ its eigenvalues For given $m \leq n$ let $\sigma \in S_m$. For $\gamma \vdash m$ let s_γ be its corresponding Schur function

Let us define $I(A, \sigma) = \int \prod A^u_{i\sigma i} du$ integrated over the unitarian group with respect to the Haar measure

Theorem 9. $I(A, \sigma) = \sum_{\eta \vdash m} \frac{s_\eta(\lambda_1 ... \lambda_n)}{s_\eta(1,1...1)} \frac{\chi_\eta(e)}{m!} \chi_\eta(\sigma)$

We give several example before the proof

1)Let A=I the unit $n \times n$ matrix
Using the formula one get
$I(I, \sigma) = \sum_{\eta \vdash m} \frac{\chi_\eta(e)\chi_\eta(\sigma)}{m!} = \delta_{e,\sigma}$
Indeed this is the second orthogonal relatione of the characters of the symmetric group

2)Let A be 2×2 matrix and λ_1, λ_2 its eigenvalues
The two partitions of 2 are the trivial (2) and the only non trivial(1,1)
Now
$s_{(2)} = \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2$
$s_{(1,1)} = \lambda_1\lambda_2$
hence $I(A, \sigma = 1, 1) = \frac{1}{2}\frac{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2}{3} + \frac{\lambda_1\lambda_2}{2}$
$I(A, \sigma = (12)) = \frac{1}{2}\frac{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2}{3} - \frac{\lambda_1\lambda_2}{2}$ Hence the determinant ie $I(A, \sigma = 1, 1) - I(A, \sigma = (12)) = \lambda_1\lambda_2$

3)Let A be a matrix of rank 1 than $I(A, \sigma) = \frac{s_m(\lambda_1,0,0...0)}{s_m(1,1,1..)}$ where λ_1 is the only non zero eigenvalue of A. Since only the one part partition Schur function supports a matrix of rank 1(Recall theorem 6)
Hence the integral doesn't depend on σ
We turn to prove the theorem
by lemma 4and Schur Weyl duality since E_A^m is $GLn(\mathbb{C})$,equivariant one can write $E_A^m = \sum a_\sigma \sigma$ summed over the symmetric group

Let us compute the coefficients a_σ

By lemma 5 $E_A^N \big|_{V_\eta \times M_\eta} = \frac{S_\eta(A)}{S_\eta(I)}$ times the identity . On the other hand by lemma 1 C_η acts as the unit on $V_\eta \times M_\eta$ and is vanished on the other isotypical components so using formula 15 to C_γ we get

$$E_A^m = \sum \frac{s_\eta(A)}{s_\eta(I)} \sum \frac{\chi_\eta(e)}{m!} \sum \chi_\eta(\sigma)\sigma \qquad (29)$$

summed over S_m and over all partitions of m . Now for $m \leqq n$ we define the zero weight space of $\otimes^m \mathbb{C}^n$ to be the span of all the tensors of type $(v_{\sigma(1)} \otimes ... \otimes v_{\sigma(N)})$ for $\sigma \epsilon S_m$

For $\eta \vdash m$ $M_\eta(0)$ be the intersection of M_η with the zero weight space

For $\gamma \vdash m$ we compute $trace E_A^m|_{M_\gamma(0) \otimes V_\gamma}$ in two ways

The first one is to substitute γ in formula 29 (it means $\sigma \Rightarrow \chi_\gamma(\sigma)$) and multiply by $dim(M_\gamma(0))$.

Now $dim(M_\gamma(0)) = \chi_\gamma(e)$ by the generalization remark after theorem 7. On the other hand we compute $trace E_A^m|_{M_\gamma(0) \otimes V_\gamma}$ explicitly using the basis of the zero weight space

$$\sum \int < A^{u \otimes m}(v_{\sigma(1)} \otimes ....v_{\sigma(m)}) C_\gamma(v_{\sigma(1)} \otimes ....v_{\sigma(m)}) > du \qquad (30)$$

As in theorem 5 one can reduce formula 30 to

$$\chi_\gamma(e) \int \sum < A^{u \otimes m}(v_1 \otimes ...v_m)(v_{\sigma(1)}... \otimes v_{\sigma(m)}) > \chi_\gamma(\sigma) du \qquad (31)$$

summed over S_m Now by use of the product rule of the scaler product (formula 10) in $\otimes^m \mathbb{C}^n$ one get

$\chi_\gamma(e) \sum \int \prod_i A^u_{i\sigma(i)} \chi_\gamma(\sigma) du$ summed over S_m

Now one get theorem 9 by equating of coefficients of χ_γ in the two ways of the trace computations

Remark Kavin Coulembier ,in a lecture given in Decin's conference at August 2011, pointed out the similarity of such integrals over the

unitarian group to those integrals in [Co Sn] at least by using the Schur Weyl duality theorem , indeed they also used Schur Wyel duality in the case of Orthogonal (Symplectic) groups but they reduced their attention to groups' matrices .See our remark in the introduction about averaging over the Orthogonal groups problems

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